THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5000 Analysis I 2015-2016
Suggested Solution to Quiz 1

1. (a) Let $A = \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\}$ and $B = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$.
Then $a < 0 < b$ for all $a \in A$ and $b \in B$, but $\sup A = \inf B = 0$.

Therefore, the statement is false.

(b) Let $A_n = [-(n+1), -n]$ for $n \in \mathbb{N}$.

A_n is bounded below, but $A = \bigcup_{n=1}^{\infty} A_n = (-\infty, -1]$, which is not bounded below.
Therefore, the statement is false.

(c) A is nonempty, so there exists at least one element a in the set A.

By definition, $\sup A$ and $\inf A$ are upper bound and lower bound of A respectively.

Therefore, $\inf A \leq a \leq \sup A$.

2. (a) Let $x \in A$.

Then, $x \in (0, x_n)$ for some $n \in \mathbb{N}$.

Therefore, $0 < x < x_n < 1$, which means A is bounded above by 1 and bounded below by 0.

(b)
- Let $\epsilon > 0$.
  Take $x = \min \left\{ \frac{\epsilon}{2}, \frac{x_1}{2} \right\}$.
  Note that $0 < x \leq \frac{x_1}{2} < x$, so $x \in (0, x_1) \subset A$.
  Also, $0 < x \leq \frac{\epsilon}{2} < \epsilon$, so $\inf A = 0$.
- Since $x_n < 1$ for all $n \in \mathbb{N}$, $\{x_n : n \in \mathbb{N}\}$ is bounded above and $\sup\{x_n : n \in \mathbb{N}\}$ exists.
  Let $u = \sup\{x_n : n \in \mathbb{N}\}$.
  Let $x \in A$, then $x \in (0, x_n)$ for some $n \in \mathbb{N}$.
  Therefore, $x < x_n \leq u$, which means u is an upper bound of A.
  Let $\epsilon > 0$, then exists $n_0 \in \mathbb{N}$ such that $u - \epsilon < x_{n_0} \leq u$.
  Take $x = \max \left\{ \frac{x_{n_0} + u - \epsilon}{2}, \frac{x_{n_0}}{2} \right\}$.
  Note that $x \geq \frac{x_{n_0}}{2} > 0$ and $\frac{x_{n_0} + u - \epsilon}{2}, \frac{x_{n_0}}{2} \leq x_{n_0}$ which implies $x < x_{n_0}$.
  Therefore, $x \in (0, x_{n_0})$.
  Also, we have
  $$u - \epsilon < \frac{x_{n_0} + u - \epsilon}{2} \leq x < x_{n_0} \leq u$$
  Therefore, $\sup A = u$.

3.
- Case 1: $x > 0$
  Then $y > x > 0$ and $\frac{\sqrt{3}}{y - x} > 0$.

By Archimedean property, there exists $n \in \mathbb{N}$ such that

$$\begin{aligned} n &> \frac{\sqrt{3}}{y-x} \\ \frac{n}{\sqrt{3}}(y-x) &> 1 \\ \frac{n}{\sqrt{3}}y &> 1 + \frac{n}{\sqrt{3}}x \end{aligned}$$

By the refined Archimedean property, since $\frac{n}{\sqrt{3}}x > 0$, there exists $m \in \mathbb{N}$ such that

$$m - 1 \leq \frac{n}{\sqrt{3}}x < m$$
$$m \leq 1 + \frac{n}{\sqrt{3}}x < \frac{n}{\sqrt{3}}y$$

Therefore, $\frac{n}{\sqrt{3}}x < m < \frac{n}{\sqrt{3}}y$ and hence

$$x < \frac{m\sqrt{3}}{n} < y.$$

- Case 2: $x \leq 0$
  $-x \leq 0$, then $-x + \sqrt{3} > 0$ and so $-\frac{x}{\sqrt{3}} + 1 > 0$.
  By Archimedean property, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} N &> -\frac{x}{\sqrt{3}} + 1 \\ x + (N-1)\sqrt{3} &> 0 \end{aligned}$$

  By using case 1, there exist p, $q \in \mathbb{N}$ such that

$$y + (N+1)\sqrt{3} > \frac{p}{q}\sqrt{3} > x + (N-1)\sqrt{3}$$

  and so

$$y > (\frac{p}{q} - N + 1)\sqrt{3}) > x.$$

  Since $\frac{p}{q} - N + 1 \in \mathbb{Q}$, there exist m, $n \in \mathbb{Z}$ such that $\frac{m}{n} = \frac{p}{q} - N + 1$. Then, we have

$$x < \frac{m\sqrt{3}}{n} < y.$$

4. (a) If I_n is a sequence of closed and bounded intervals which is nested, i.e.

$$I_{n+1} \subset I_n \qquad \text{for all } n \in \mathbb{N},$$

then there exists $p \in \mathbb{R}$ such that $p \in I_n$ for all $n \in \mathbb{N}$.

(b) (i) Note that I_{n_r} is a subsequence of I_n, so I_{n_r} is a sequence of closed and bounded intervals. Also $n_r < n_{r+1}$, so

$$I_{n_{r+1}} \subset I_{n_{r+1}-1} \subset I_{n_{r+1}-2} \subset \cdots \subset I_{n_r}$$

which implies that I_{n_r} is a nested sequence.

(ii) Note that $n_r \geq r$ for all $r \in \mathbb{N}$, so $I_{n_r} \subset I_n$.
Suppose that ξI_{n_r} for all $r \in \mathbb{N}$.
Let $j \in \mathbb{N}$, then $n_j \geq j$ and so $\xi \in I_{n_j} \subset I_j$.
Therefore, $\xi \in I_j$ for all $j \in \mathbb{N}$.